Entropy in Noncommutative Geometry

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Noncommutative geometry: spectral triples

Motivation:

- Generalise geometry to quantum spaces,
- Encode geometry of space through operators on a Hilbert space,
- Provide unified framework linking geometry with physics, specifically quantum systems and gauge theories.

Spectral triple:

$$(\mathcal{A}, \mathcal{H}, D)$$

- Hilbert space $\mathcal{H}$,
- Unital $*$-algebra $\mathcal{A}$ represented as bounded operators on $\mathcal{H}$,
- Operator D on $\mathcal{H}$, which is self-adjoint, has compact resolvent and such that $[D, a]$ gives a bounded operator, see, e.g., [2, 3].

Second quantisation: Fock spaces

The fermionic Fock space is

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \qquad \text{for } \mathcal{H}_j = \mathcal{H} \wedge \dots \wedge \mathcal{H} \ (j \text{ times}).$$

We can use D to change the inner product on $\mathcal{H}$ (and $\mathcal{F}$): $\langle u|v \rangle_I = g(u, v) + ig(Iu, v)$ for $I = i\,\mathrm{sign}(D)$.

Entropy

The **von Neumann entropy** of state $\phi(\cdot) = \mathrm{Tr}(\rho\,\cdot\,)$ is

$$S(\phi) := -\mathrm{Tr}(\rho \log \rho) \qquad (1)$$

for ρ a density matrix on the Hilbert space.

KMS-states

We are interested in equilibrium states $\longrightarrow$ KMS$_\beta$-states: ϕ such that

$$\phi(a\sigma_t(b))|_{t=i\beta} = \phi(ba)$$

for all operators a, b and time evolution $\sigma_t(X) \sim e^{itH} X e^{-itH}$.

References

[1] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom. Entropy and the Spectral Action. *Commun. Math. Phys.*, 373(2):457–471.

[2] A. Connes. *Noncommutative Geometry*. Academic press.

[3] W. D. van Suijlekom. *Noncommutative Geometry and Particle Physics*. Mathematical Physics Studies. Springer Nature Switzerland.

Example: two-point space

$$F = \overset{1}{\bullet} \overset{r}{\longleftrightarrow} \overset{2}{\bullet}$$

$$\Updownarrow$$

$$\left(C^\infty(F) = \mathbb{C}^2, \mathcal{H} = \mathbb{C}^2, D = \begin{pmatrix} 0 & 1/r \\ 1/r & 0 \end{pmatrix} \right)$$

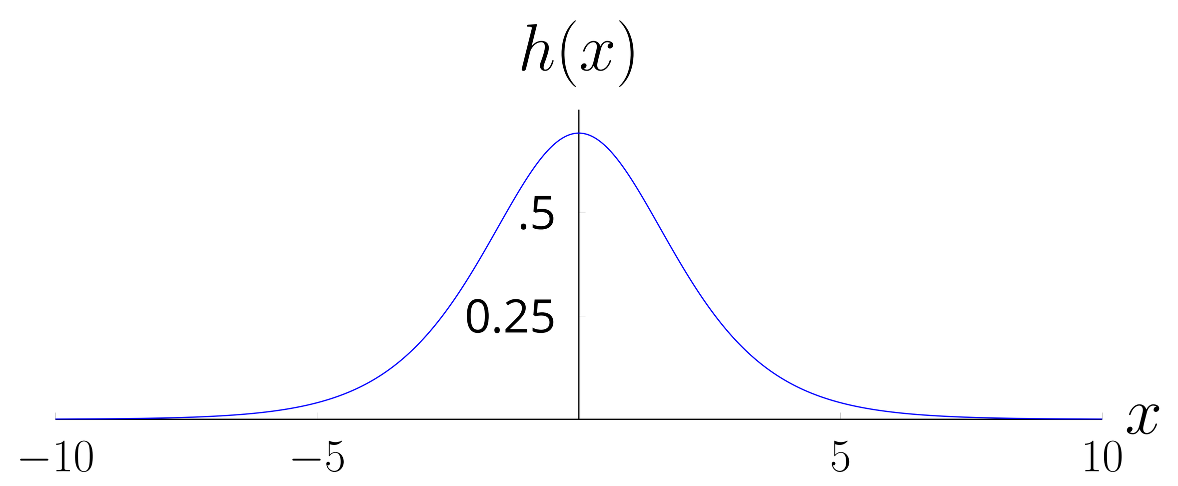
We want the entropy on Fock space $\mathcal{F} \cong \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$.

- Denote by
$$\Lambda e^{-\beta|D|}(u_1 \wedge \dots \wedge u_k) = \left(e^{-\beta|D|}u_1\right) \wedge \dots \wedge \left(e^{-\beta|D|}u_k\right)$$
the lift of operator $e^{-\beta|D|}$ on $\mathcal{H}$ to $\mathcal{F}$.
- Fix $H = -\frac{1}{\beta}\log\left(\Lambda e^{-\beta|D|}\right)$, i.e.,
$$H(u_1 \wedge \dots \wedge u_k) = -\sum_{j=1}^{k} u_1 \wedge \dots \wedge (|D|u_j) \wedge \dots \wedge u_k.$$
- KMS$_\beta$-states are unique:
$$\psi_\beta(\cdot) = \mathrm{Tr}_{\mathcal{F}}(\rho_\beta\,\cdot\,) \quad \text{for} \quad \rho_\beta = \frac{1}{\mathrm{Tr}_{\mathcal{F}}(\Lambda e^{-\beta|D|})}\Lambda e^{-\beta|D|}.$$

Entropy for two-point space

The entropy of the KMS$_\beta$-state is (see also [1])

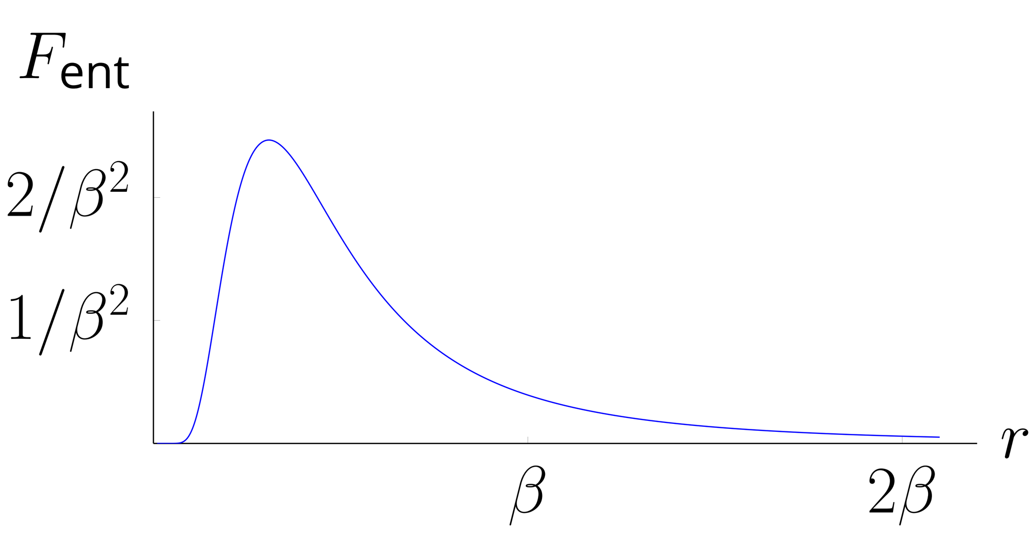
$$S(\psi_\beta) = 2h(\beta/r) = 2\frac{\beta/r}{1 + e^{\beta/r}} + 2\log\left(1 + e^{-\beta/r}\right). \qquad (2)$$



Entropic force and emergence

Maximising r gives maximal entropy, the entropic force enforces this:

$$F_{\mathrm{ent}}(r) := \frac{1}{\beta}\partial_r S(\rho_\beta) = \frac{\beta/2r^3}{\cosh^2(\beta/2r)}.$$



Interpret the corresponding tendency to have r increase as the '**emergence** of two points from a single one'.